Lecture 2

Sets and relations

Theorem 1. If X and Y are sets and $F: X \to Y$ is bijection, then $F^{-1}: Y \to X$ is also one-to-one and onto.

Proof. <u>F^{-1} is one-to-one:</u> Suppose $y_1, y_2 \in Y$ and $F^{-1}(y_1) = F^{-1}(y_2)$. Let $x = F^{-1}(y_1) = F^{-1}(y_2)$. Then $x \in X$ and by definition of F^{-1}:

$$F(x) = y_1 \quad \text{since } x = F^{-1}(y_1)$$

$$F(x) = y_2 \quad \text{since } x = F^{-2}(y_2).$$

Consequently $y_1 = y_2$, because each is equal to $F(x)$.

F^{-1} is onto: Suppose $x \in X$.
We must show that there exists an element in Y such that $F^{-1}(y) = x$.

Let $y = F(x)$. Then $y \in Y$ and by definition of F^{-1} we have that
$F^{-1}(y) = x$. ▶

Remind. Suppose that $F: X \to Y$ one-to-one correspondence (bijection).
Then there is function $F^{-1}: Y \to X$ that is defined as follows:

$$\forall y \in Y \quad F^{-1}(y) = x \iff y = F(x)$$

(a unique element $x \in X$, such that $y = F(x)$)

Def. Sets A and B are called equivalent if there exists a bijection $f : A \to B$.

We denote equivalent sets as $A \sim B$.

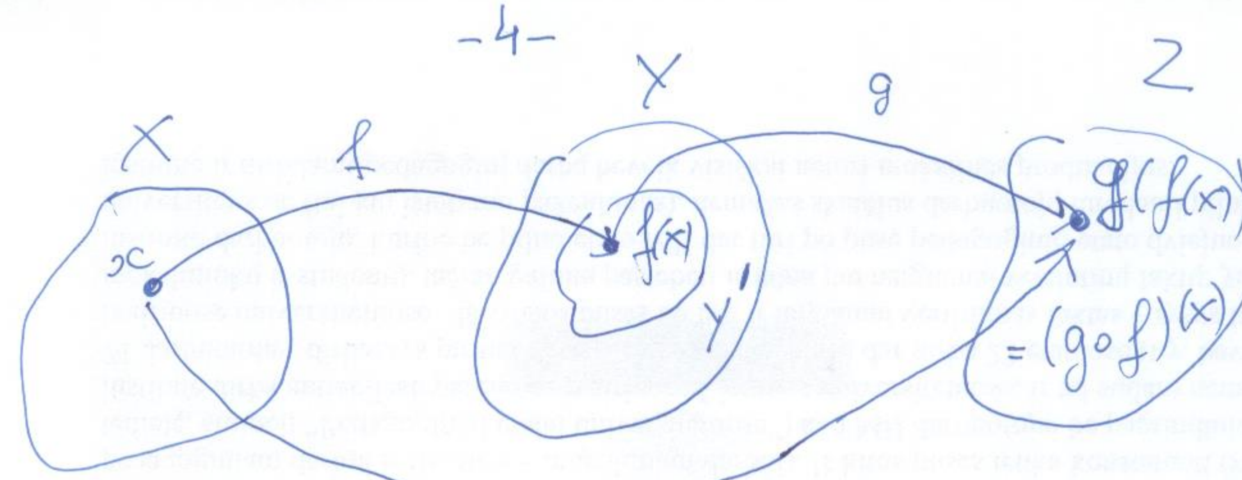
Composition of Functions

Let $f : X \to Y$ and $g : Y' \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f : X \to Z$ as follows

$$\boxed{(g \circ f)(x) = g(f(x)) \quad \forall x \in X}$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x".

$g \circ f$ is called the composition of f and g



Example. $f : \mathbb{Z} \to \mathbb{Z}$ $\quad f(n) = n+1$

$g : \mathbb{Z} \to \mathbb{Z}$ $\quad g(n) = n^2$

$\forall n \in \mathbb{Z}$.

$$(g \circ f)(n) = g(f(n)) = g(n+1) = (n+1)^2$$

$$(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1.$$

$$g \circ f \neq f \circ g$$

The composition of functions is not a commutative operation (in general).

$AB \neq BA$ for matrices.

Ex. 1 Find for $f: X \to Y$ compositions

$f \circ I_X$ and $I_Y \circ f$,

here I_X is the identity function

$I_X: X \to X$ $\quad (I_X(x) = x)$.
$\forall x \in X$.

Ex. 2. Find for $f: X \to Y$ (bijection)

$f^{-1} \circ f$ and $f \circ f^{-1}$

Ex. 3 (hard). If $f: X \to Y$ and

$g: Y \to Z$ are both one-to-one

functions, then $g \circ f$ is one-to-one

(see analys[is]
Example 7.3.5

Relations

Definition. Let A and B be sets.

A relation R from A to B is a subset of $A \times B$ (binary relation)

Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written $x R y$, iff (x, y) is in R.

The set A is called the domain of R and the set B is called its co-domain.

$x R y$ means that $(x, y) \in R$.

The notation

$x \not R y$ means that x is not related to y by R

$x \not R y$ means that $(x, y) \notin R$.

Example 1. Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. We define a relation R from A to B: given $(x, y) \in A \times B$

$(x, y) \in R$ means that $\frac{x-y}{2}$ is an integer.

a) Which ordered pairs are in $A \times B$

b) Which — " — are in R.

c) What are domain and co-domain of R?

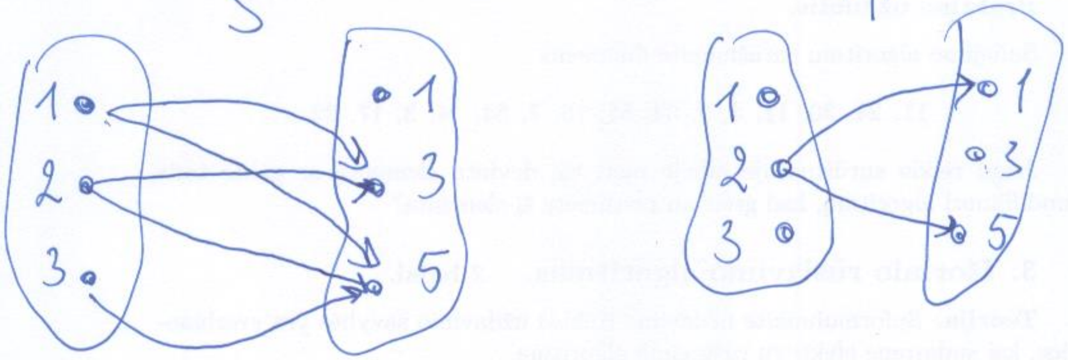
Example 2 Arrow Diagrams of Relations

Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B. For every $(x, y) \in A \times B$

$(x, y) \in S$ means that $x < y$

("less than" relation)

$$T = \{(2,1), (2,5)\}$$



Functions can be defined as relations.

<u>Def</u> A function F from a set A to a set B is a relation with domain A and co-domain B that satisfies the following two properties

1. For $\forall x \in A$ there is $y \in B$ such that $(x, y) \in F$.

2. For all elements x in A and y and z in B,

if $(x, y) \in F$ and $(x, z) \in F$,

then $y = z$.

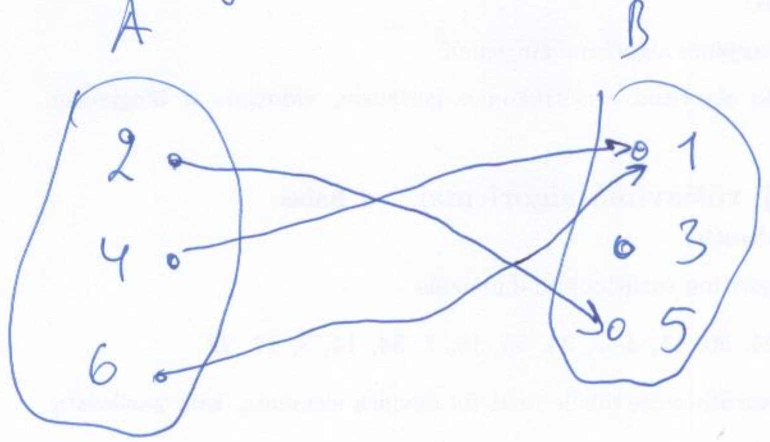
Notation: $F(x)$ (F of x).

<u>Example</u>. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R, S, and T are functions from A to B:

a) $R = \{(2,5), (4,1), (4,3), (6,5)\}$

b) $\forall (x,y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$

c) T is defined by the arrow diagram



In mathematics domain and co-domain of functions and relations can be defined in a more strict way:

- the domain of a relation (function) is the set of elements (values) that we a allowed to plug into our relation (function)

- The range of a relation (function) is the set of values that the relation relates to (function assumes).

Example. Let the relation S is defined as

$$S = \{(a_1, b_1), (a_1, b_2), (a_2, b_3)\}$$

The domain of definition is equal to

$$\mathcal{D}(S) = \{a_1, a_2\}$$

The range is equal to

$$\mathcal{R}(S) = \{b_1, b_2, b_3\}.$$

Basic properties of Binary Relations

Let's consider relations $R \subset A^2 = A \times A$

1. R is said ~~to~~ be a <u>reflexive</u> <u>relation</u> in A if

$$\forall a \in A \Rightarrow (a, a) \in R.$$

Example 1 $I_A \subset A^2$ is reflexive.
(I_A is the identity relation)

<u>Example 2.</u> $A = \{1, 2, 3\}$

~~R~~ $R_1 = \{(1,1), (1,2), (2,2), (2,3), (3,3)\} \subset A \times A$ is reflexive

<u>Example 3</u> $R_2 = \{(1,1), (1,2), (2,2), (2,3), (3,2)\}$ is <u>not</u> reflexive

Theorem. $R \subset A \times A$ is reflexive iff $I_A \subset R$

$$R \text{ is reflexive} \Leftrightarrow I_A \subset R.$$

Proof. $\Rightarrow$ $I_A = \{(a, a),\ \forall a \in A\}$

If R is reflexive then

$$(a, a) \in R \quad \forall a \in A$$

thus $I_A \subset R$.

$\Leftarrow$ if $I_A \subset R$ then

$$(a, a) \in R \quad \forall a \in A$$

thus R is reflexive.

Definition. $R \subset A \times A$ is called a *symmetric relation* if for

$$(a, b) \in R \Rightarrow (b, a) \in R.$$

Def. Relation $R^{-1} \subset A^2$ is called an <u>inverse relation</u> ~~to~~ the relation $R \subset A^2$:

$$R^{-1} = \{(a, b) : (b, a) \in R\}.$$

Remark.

$$(R^{-1})^{-1} = R.$$

Give a proof of this property.

Property.

symmetric relation $\Leftrightarrow R = R^{-1}$

<u>Proof</u> $\Rightarrow$ If R symmetric, then

$$\forall (a, b) \in R \Rightarrow (b, a) \in R$$
$$\Rightarrow (a, b) \in R^{-1}$$

$$\forall (a, b) \in R^{-1} \Rightarrow (b, a) \in R \overset{symm}{\Rightarrow} (a, b) \in R.$$

Make a proof for $\Leftarrow$ part

Operations on Relations

Good news: the union, the intersection, the difference and the complement of relations are defined as the corresponding set operations.

Examples. $\mathbb{Z}$ is the set of integers.

$\varphi_j \subset \mathbb{Z} \times \mathbb{Z}$, $j = 1, 2, 3$ are relations

$$\varphi_1 = \{(a, b) : a \geq b\}$$

$$\varphi_2 = \{(a, b) : a > b\}$$

$$\varphi_3 = \{(a, b) : a < b\}$$

1. $\varphi_2 \subset \varphi_1$

2. $\varphi_1 \cup \varphi_2 = \varphi_1$

3. $\varphi_1 \cap \varphi_2 = \varphi_2$

4. $\varphi_3^c = \varphi_1$ $\quad (\overline{\varphi_3} = \varphi_1)$

5 $\varphi_1 \setminus \varphi_2 = I_{\mathbb{Z}}$ $\quad (\varphi_1 - \varphi_2 = I_{\mathbb{Z}})$

Prove these statements by using the definitions of operations on sets.

Def. Relation $R \subset A \times A$ is a transitive relation if $\forall a, b, c \in A$ such that $(a, b) \in R$, and $(b, c) \in R \Rightarrow$ $\boxed{(a, c) \in R.}$

Example 1 $A = \{1, 2, 3\}$

$R = \{(1, 2), (1, 3)\}$

R is a transitive relation.

Example 2 $A = \{1, 3, 5\}$

$R = \{(1, 3), (3, 5)\}$

R is not a transitive relation.

Def. (Composition of relations)

Let A, B and C are sets and we have relations

$\varphi \subset A \times B, \ \psi \subset B \times C.$

Then we define a new relation

(a composition $\varphi \circ \psi$ of relations φ, ψ)

$\varphi \circ \omega \in A \times C$

$\varphi \circ \psi = \{ (a, c) : \exists b \in B \neq (a, b) \in \varphi$

and $(b, c) \in \psi \}$.

Let's consider a case $A = B = C$.

Examples.

$\varphi_1 \circ \varphi_2 = \varphi_2$, $\varphi_2 \circ \varphi_1 = \varphi_2$

$\varphi_1 \circ \varphi_3 = \varphi_3 \circ \varphi_1 = \varphi_2 \circ \varphi_3 = U_Z = \mathbb{Z}^2$

Remark. In general case

$$\varphi \circ \psi \neq \psi \circ \varphi$$

Give your examples.

Def. The power of relation $R \subset A^2$ is the following composition of relations

$R^0 = I_A$, $R^1 = R$, $R^2 = R \circ R$,

$R^n = R^{n-1} \circ R$.

<u>Theorem</u> Relation R is a transitive relation iff (if and only if)

$$R \circ R \subset R.$$

<u>Proof</u> $\Rightarrow$

If R is transitive $\Rightarrow$ for all $(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R$.

Let's assume that $(a, c) \in R \circ R$, then there exits $b \in A$ such that $(a, b) \in R$ and $(b, c) \in R \overset{\text{transitive}}{\Longrightarrow} (a, c) \in R$.

Thus $R \circ R \subset R$.

Prove $\Rightarrow$ part of the theorem.